A Time-Frequency Analysis of Oil Price Data

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Purpose and summary

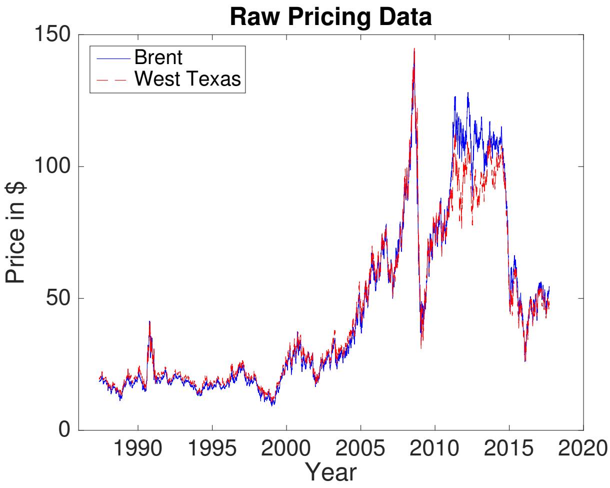
Purpose:

- Challenge: to understand oil price dynamics in the long run, but also on short scales, in particular, for the recent period (since 2014).
- Idea: to adapt and exploit elaborated tools developed for turbulence data.

Summary:

- Oil price data have a rich multiscale structure that may vary over time.
- The monitoring of these variations shows regime switches.
- The quantitative analysis is carried out by a wavelet decomposition method.

The data set



Oil price data from 1987 to 2017 for West Texas (red dashed line) and Brent (solid blue line).

Early history of price modeling

1900: Louis Bachelier "Théorie de la Spéculation":
- Modeling of prices and pricing of options.
- Analysis of Brownian motion (predates Einstein's 1905 works).

The price changes over different time intervals $[t_n, t_{n+1}]$ are independent, Gaussian, zero-mean, and with variance proportional to $t_{n+1} - t_n$.

Increments of Brownian motion model "absolute" price changes, essentially:

$$P(t_{n+1}) - P(t_n) = \sigma(B(t_{n+1}) - B(t_n)).$$

Standard price model

1960s: Paul A. Samuelson:

Increments of Brownian motion model "relative" price changes, essentially:

$$\frac{P(t_{n+1}) - P(t_n)}{P(t_n)} = \sigma(B(t_{n+1}) - B(t_n)),$$

with, in addition, possibly a deterministic drift.

Standard price model

- Price $P(t)$ = a drift $d(t)$ + a diffusion, that can be expressed in terms of Brownian motion $B(t)$ and volatility σ_t :

$$\frac{dP(t)}{P(t)} = d(t)dt + \sigma_t dB(t)$$

- The Brownian motion B is a Gaussian process with independent and stationary increments:

$$\mathbb{E}\big[(B(t+\Delta t) - B(t))^2\big] = |\Delta t|$$

It is self-similar:

$$B(at) \overset{dist.}{\sim} a^{1/2}B(t) \quad \text{for any } a > 0$$

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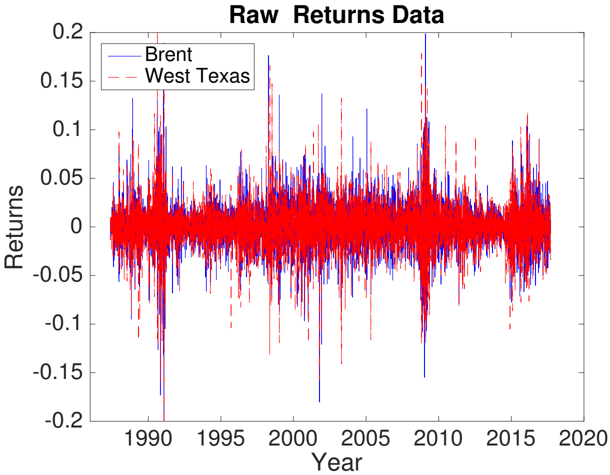
- The volatility is the most important ingredient of the standard model.
  - When $\sigma_t \equiv \sigma$ and $d(t) \equiv d$, this is the Black–Scholes model (1973).
  - When $\sigma_t \equiv \sigma(t, P(t))$, this is the local volatility model (Dupire, 1994; Derman and Kani, 1994).
  - When σ_t is a stochastic process, this is the stochastic volatility model (Hull and White, 1987; Heston, 1993).



Returns for West Texas (red dashed line) and Brent (solid blue line).

$$R_{n+1} = \frac{P(t_{n+1}) - P(t_n)}{P(t_n)}, \quad t_n = n\Delta t$$

Modeling of prices: An alternative approach

1960s: Benoit Mandelbrot:

Increments of fractional Brownian motion model "relative" price changes, essentially:

$$R_{n+1} = \frac{P(t_{n+1}) - P(t_n)}{P(t_n)} = \sigma(B_H(t_{n+1}) - B_H(t_n)).$$

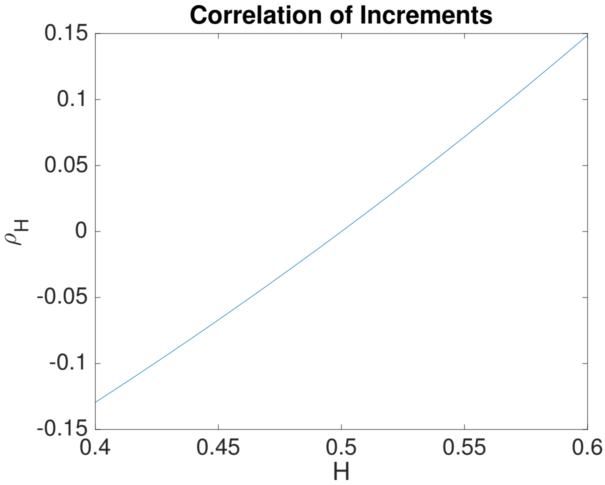
$\rightarrow$ Returns have "memory":

$$\rho_H = \frac{\mathbb{E}[R_{n+1} R_n]}{\mathbb{E}[R_n^2]} = 1 - 2^{2H-1}.$$

Thus in general:

$$\begin{aligned} \mathbb{E}[R_{n+1} \mid R_n] &= \rho_H R_n \\ &\neq \mathbb{E}[R_{n+1} \mid R_n, R_{n-1}]. \end{aligned}$$

Hurst sensitivity of return correlation



$$\rho_H = \frac{\mathbb{E}[R_{n+1}R_n]}{\mathbb{E}[R_n^2]} = 1 - 2^{2H-1}.$$

Fractional price model

- Price $P(t)$:

$$\frac{dP(t)}{P(t)} = d(t)dt + \sigma_t dB_H(t)$$

  where B_H is a fractional Brownian process with Hurst index H.
- B_H is a Gaussian process with dependent and stationary increments:

$$\mathbb{E}\big[(B_H(t+\Delta t) - B_H(t))^2\big] = |\Delta t|^{2H}$$

- Properties:
  - It is self-similar:

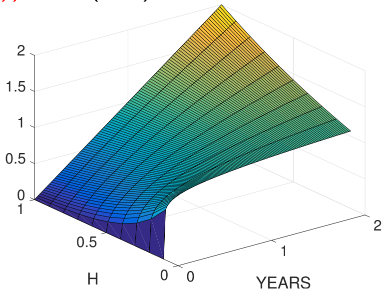
$$B_H(at) \overset{dist.}{\sim} a^H B_H(t) \quad \text{for any } a > 0$$

  - If $H = 1/2$: it has uncorrelated increments (standard Brownian motion).
  - If $H < 1/2$: it has negatively correlated increments (anti-persistence). Trajectories are rough (but continuous).
  - If $H > 1/2$: it has positively correlated increments (persistence). Trajectories are smooth (but not differentiable).

Radical change of perspective: The Hurst coefficient and volatility are the fundamental quantities.

The Hurst coefficient governs the scaling of volatility with the time lag.

Example: If we condition "volatility" to be one at time lag 1 (say in annualized units), then St.dev$(\sigma B(t))$, $t \in (0,2), \sigma = 1$:



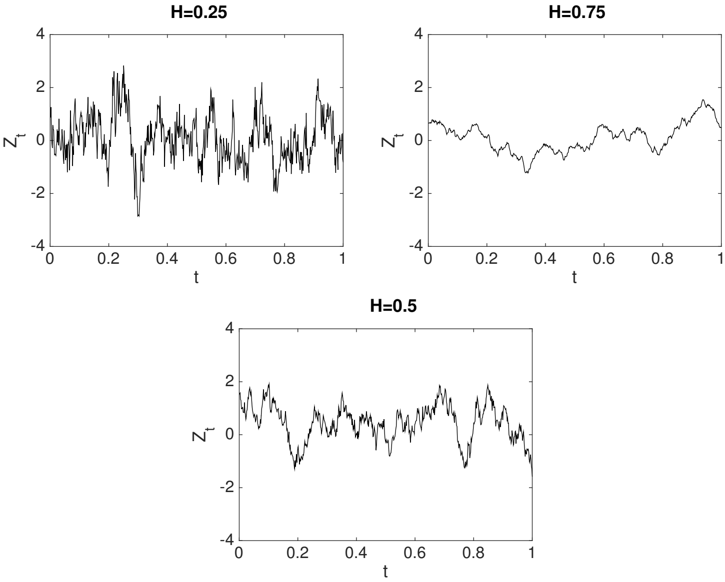
Classic case:

$H = 1/2$: independent increments.

Limit cases:

$H \to 1$: Increments equal: $\Delta B_n = \Delta B_{n-1}$.

$H \to 0$: Increments alternate in sign: $\Delta B_n = -\Delta B_{n-1}$.



Ornstein–Uhlenbeck processes

$$dZ(t) = -Z(t)dt + dB_H(t)$$

History

Fractional processes:

- Kolmogorov (1940): turbulence (turbulent flow composed by "eddies" of different sizes).
- Hurst (1951): hydrology (fluctuations of the water level in the Nile River).
- Mandelbrot and Van Ness (1968): finance.
- Comte and Renault (1998): stochastic volatility.
- Gatheral et al (2014): rough stochastic volatility.

Multi-fractional price model

→ Motivated by the data, let increments of multi-fractional Brownian motion model "relative" price changes, essentially:

$$R_{n+1} = \frac{P(t_{n+1}) - P(t_n)}{P(t_n)} = \sigma_n(B_{H_n}(t_{n+1}) - B_{H_n}(t_n)).$$

- Price $P(t)$:

$$\frac{dP(t)}{P(t)} = d(t)dt + dB_{H,\sigma}(t)$$

  where $B_{H,\sigma}$ is a multi-fractional process (H_t and σ_t are time-dependent) (Lévy-Véhel 1995).
- If $H_t \equiv H$ and $\sigma_t \equiv 1$, then $B_{H,\sigma} = B_H$ fractional Brownian motion.

→ Some issues:

- Rapid Monte-Carlo simulation of price "paths".
- Estimation of the parameters.
  ↪ use of wavelets.

Wavelets

- Main context: Signals may have frequency content that varies with time. Ex: speech.
- A Fourier decomposition gives the "global" frequency decomposition.
- A wavelet decomposition gives a local characterization of the frequency contents.

$\rightarrow$ To detect changes in the multiscale character of the prices the wavelets are useful.

Cf: Meyer 1984, Mallat, Daubechies...

Joint estimation of Hurst and volatility

- Use wavelet decomposition to localize in time and frequency the different components of the time series.
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- Remark: Many methods for Hurst parameter estimation:
  - Box-count estimator (Hall and Wood, 1993).
  - Variogram estimator (Constantine and Hall, 1994).
  - Level crossing estimator (Feuerverger, Hall and Wood, 1994).
  - Variation estimators (multiscale moments).
  - First spectral and wavelets estimators (Chan, Hall and Poskitt, 1995).

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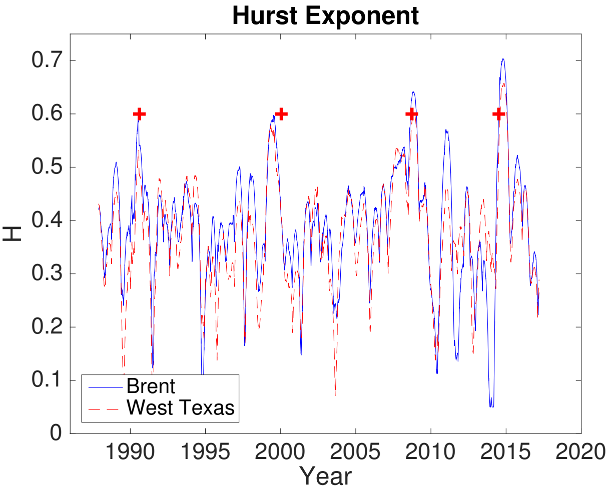
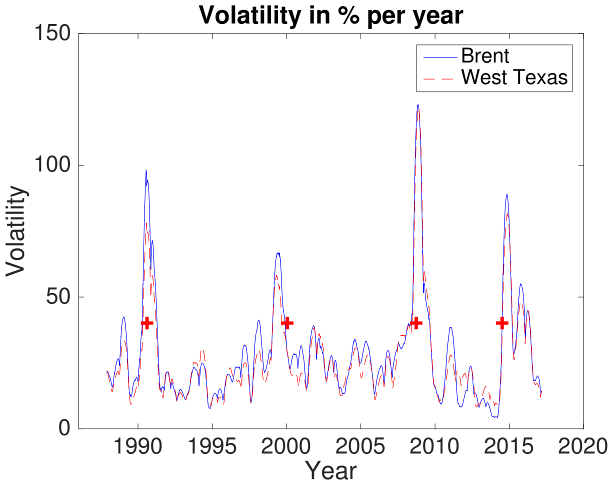
  - First spectral and wavelets estimators (Chan, Hall and Poskitt, 1995).
- Statistical characterization of the local Hurst and volatility estimator.



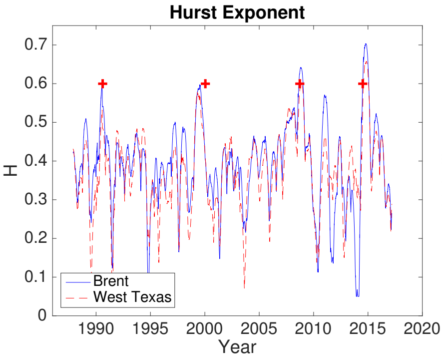
Estimated Hurst exponents H_t.

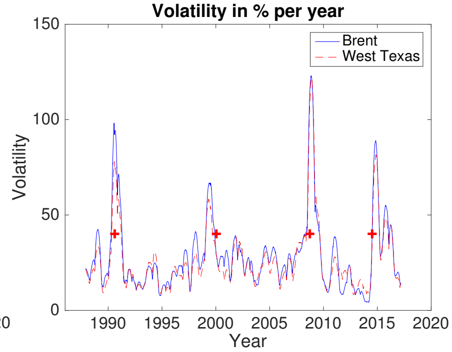
- There are four periods with a relatively high Hurst exponent, which can be related to four events, marked with crosses.



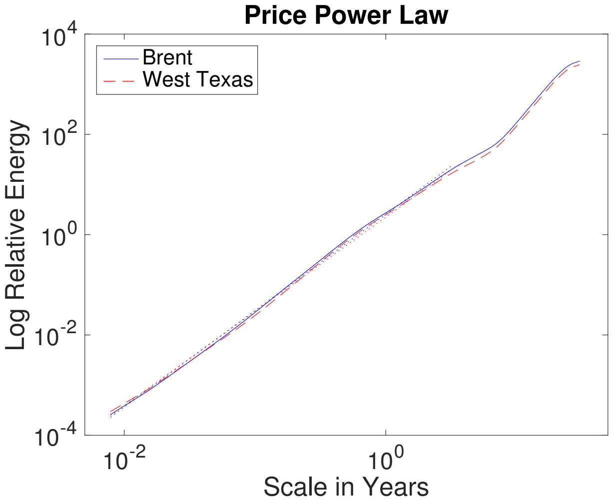
Local volatility estimates relative to the annual time scale σ_t.

- There are four periods with relatively high volatility, which can be related to four events, marked with crosses.
- Volatility is stable except: four special periods + period 2010-2014 (decaying volatility).



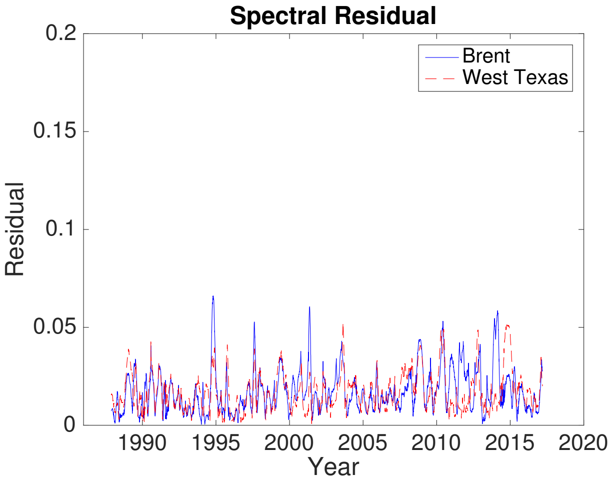


- **August 1990**: Iraq's invasion of Kuwait; it initiated a period with high volatility and a high Hurst exponent.
- **January 2000**: fear of the Y2K bug (?), which never occurred; it ended a period with relatively high volatility and Hurst exponent.
- **September 2008**: bankruptcy of Lehman Brothers; it initiated a period with very high volatility and a high Hurst exponent.
- **July 2014**: liquidation of oil-linked derivatives by fund managers; it initiated a period with a very high Hurst exponent and high volatility.



The "global power law" for West Texas data (red dashed line) and Brent data (blue solid line).

- A global power law (with $H = .47$) is consistent with a situation in which the Hurst exponent and volatility vary over subsegments.



Spectral misfits for the West Texas data (red dashed line) and the Brent data (solid blue line).

- The spectral misfits are low and statistically homogeneous with respect to time.

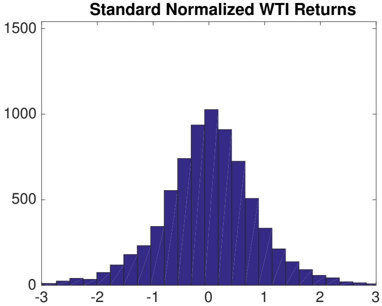
Are returns Gaussian?

- Standard normalized returns:

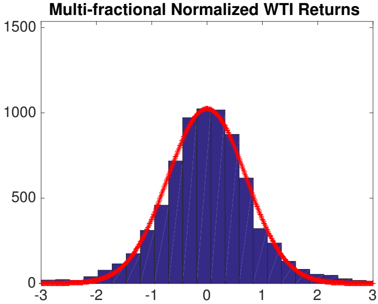
$$R_n^{(s)} = \frac{\log(P(t_n)) - \log(P(t_{n-1}))}{\sigma|\Delta t|^{1/2}}.$$

- Multi-fractional normalized returns:

$$R_n^{(m)} = \frac{\log(P(t_n)) - \log(P(t_{n-1}))}{\sigma_n|\Delta t|^{H_n}}.$$

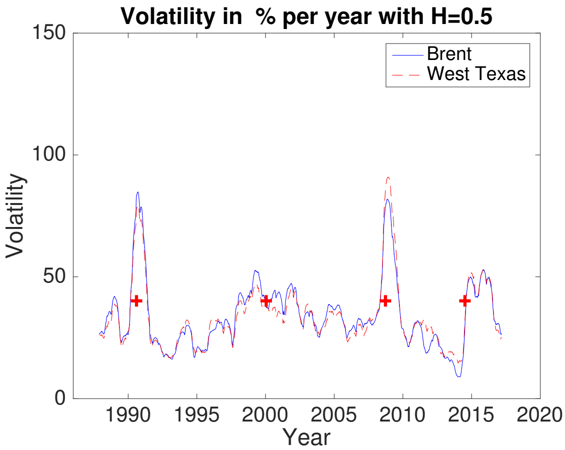


Standard normalized returns



Multi-fractional normalized returns

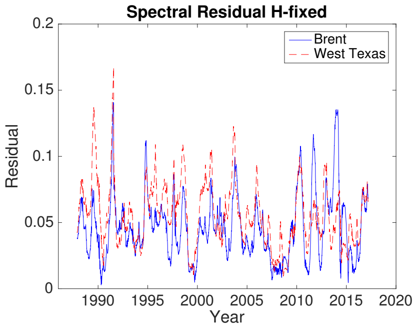
Comparison with standard model

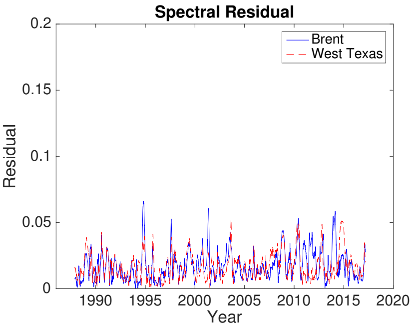


Estimated volatilities when we condition on $H = 1/2$ to enforce uncorrelated returns.

- The four special periods do not appear so clearly; beyond these special periods, the standard volatility experiences somewhat strong variations.

Comparison with standard model





Spectral misfits

Left when we condition on $H = 1/2$ to enforce uncorrelated returns.

Right with the multi-fractional model.

- With H fixed, the spectral misfits are relatively high; they also vary significantly during the special periods.

Conclusions so far

- Oil data contain multiscale fluctuations different from (log) normal diffusions.
- Special periods with $H_t > 1/2$ can be identified.
- Standard volatility estimates are not appropriate during the special periods when $H_t > 1/2$.

On classic and real markets

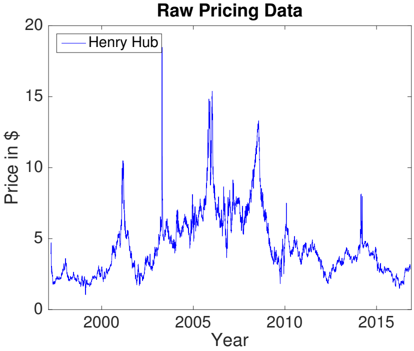
$\rightarrow$ In "classic" financial markets (with no arbitrage), the conditional expectation of the returns are zero.

$\rightarrow$ In "real" markets, we see deviations from the "ideal classical" context.

$\rightarrow$ For $H \neq 1/2$, the market is not efficient and pricing and hedging become challenging.

In what follows: different markets.

Natural gas price



Henry Hub natural gas spot price.

Some events:

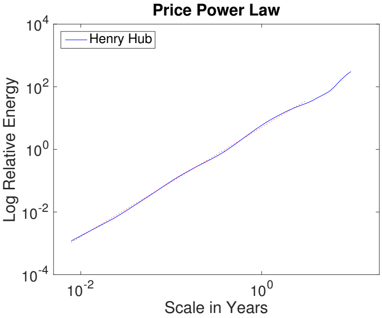
– **2001**: California energy crisis;

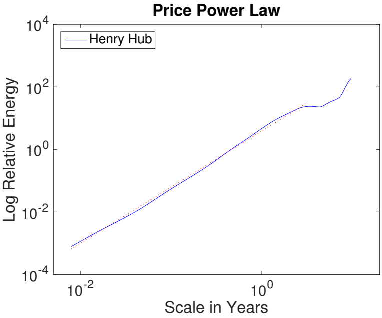
– **2003**: cold winter;

– end of **2005**: hurricanes Katrina and Rita and volatile weather;

– **2008**: the price increase corresponding to the high oil price.

Natural gas price

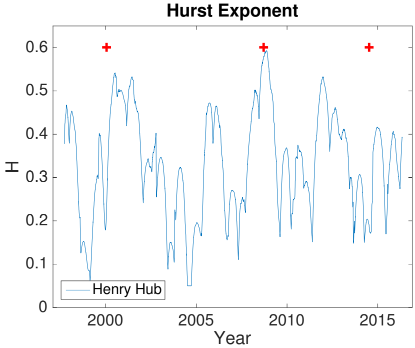


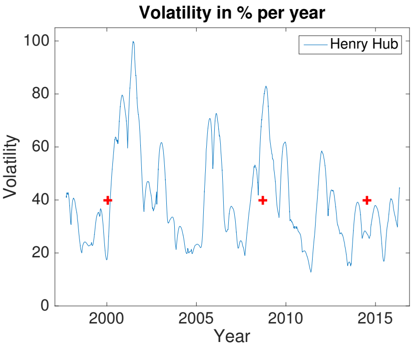


Left: Scale spectrum for Henry Hub natural gas for the period **1997–2009**. The estimated Hurst exponent is $H = .37$. The estimated volatility is $\sigma = 49\%$. The red dotted line is the fitted spectrum and fits the data well up to an outer scale of more than one year.

Right: Scale spectrum for Henry Hub natural gas for the period **2009–2016**. The estimated Hurst exponent is $H = .40$. The estimated volatility is $\sigma = 43\%$.

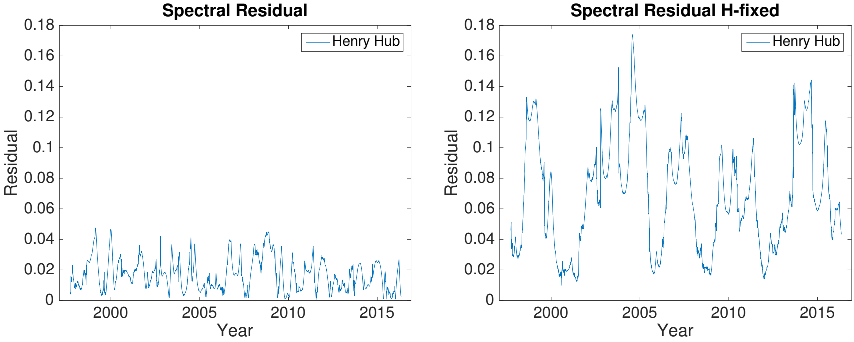
Natural gas price





Left: Estimated Hurst exponents H_t for the Henry Hub natural gas spot price.
Right: Estimated volatility σ_t for the Henry Hub natural gas spot price.
Crosses:
January 2000: Y2K bug (?); **September 2008**: bankruptcy of Lehman Brothers; **July 2014**: liquidation of oil-linked derivatives by fund managers.

Natural gas price



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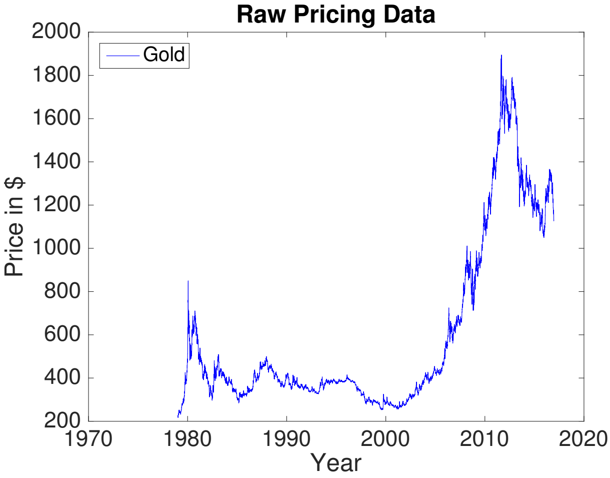
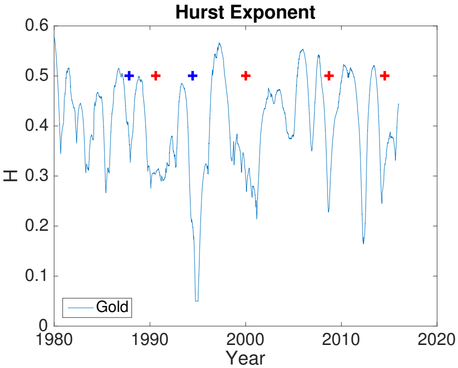
Gold



Daily gold prices.

Gold multi-fractal character



Daily gold prices.

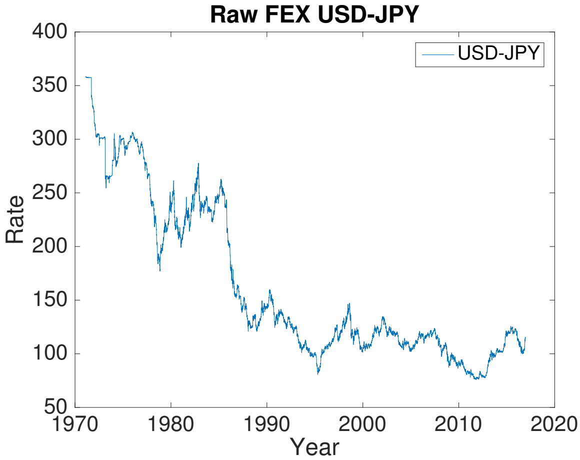
Blue crosses:

- Black Monday, October 19, 1987.
- Mexican peso crisis, December 1994.

Red crosses as above, but now associated with a rough period.

Foreign exchange

Consider the daily closing of yen-per-dollar price:



The price process corresponding to the daily closing of yen-per-dollar price.

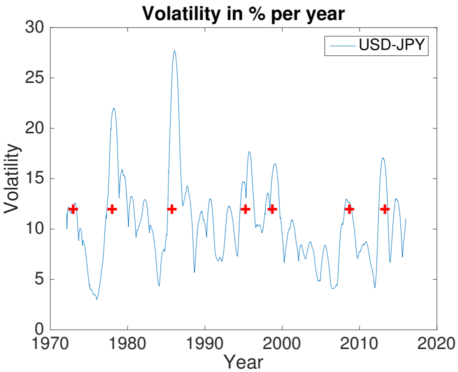
- Does the exchange rate possess a multiscale structure?

USD–JPY scale spectrum

Scale spectrum for the full data set.

$\rightarrow$ Efficiency on the grand scale with estimated $H = .51$.

USD–JPY volatility

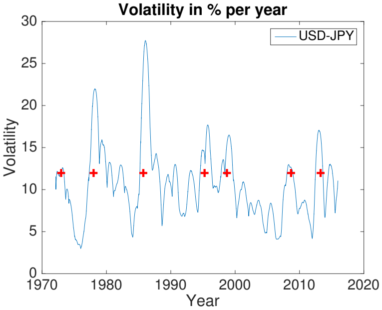


– **January 1973**: A series of events led to the first oil crisis that hit in October 1973;

– **January 1978**: A series of events led to the Iranian revolution of 1979 and the second oil crisis;

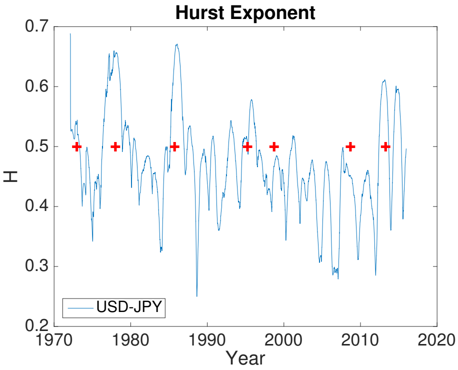
– **September 1985**: The *Plaza Accord.* This was a signed agreement between major nations affirming that the dollar was overvalued.

USD–JPY volatility



– **April 1995**: The yen briefly hit a peak of under 80 yen per dollar after US–Japanese trade frictions sparked heavy selling of the dollar.
– **October 1998**: Near collapse of the hedge fund Long-Term Capital Management.
– **September 2008**: Lehman Brothers collapsed.
– **April 2013**: The Bank of Japan announced the expansion of its Asset Purchase program.

USD–JPY Hurst



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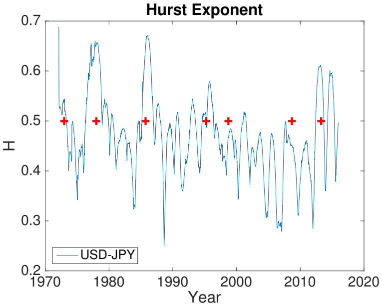

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Some further issues

- "High-frequency" data.
- Periodicities.
- Cross-commodity correlations and derivatives.
- Interest rate.
- Consequences for speculation and risk management.

Final remarks

- Multi-fractal behavior can be observed in various markets.
- We have developed a theory for the performance of the estimator.
  $\rightarrow$ The Haar wavelets are partly superior.
- Multi-fractal modeling means a departure from classic financial modeling.
- Power-law modeling *potentially important* for regime shift detection, prediction, pricing, hedging,...
- Viewing the market and prices through the lens of both roughness and magnitude scaling (H, σ) gives "complementary" (economic) insight about the market.
- Claim: As in physics, H is a defining parameter. $H < 1/2$ gives a modification of the efficient market situation, while $H > 1/2$ changes the problem in a more fundamental way.

Appendix: The case of dyadic Haar wavelets

Denote the approximation coefficients at level zero (the data) by:

$$\boldsymbol{X} = (a_0(1), a_0(2), ..., a_0(2^M)).$$

Then, at the scale j (corresponding to frequence 2^{-j}), define the approximation and difference coefficients by:

$$\begin{aligned} a_j(n) &= \frac{1}{\sqrt{2}}(a_{j-1}(2n) + a_{j-1}(2n-1)) \\ d_j(n) &= \frac{1}{\sqrt{2}}(a_{j-1}(2n) - a_{j-1}(2n-1)) \ , \quad \text{for} \quad n = 1, 2, ..., 2^{M-j} \end{aligned}$$

for $j = 1, ..., M$.

Appendix: Coefficients from the continuum

If $a_0(n) = \int_{n-1}^{n} Y(t)dt$, then the detail coefficients at level j can alternatively be expressed as:

$$d_j(n) = 2^{-j/2} \int_{-\infty}^{\infty} \psi(t2^{-j} - n) Y(t) dt$$

for Y, the (quasi-continuous) data, and with the mother wavelet defined by

$$\psi(x) = \begin{cases} -1 & \text{if} \quad -1 \le x < -1/2 \\ 1 & \text{if} \quad -1/2 \le x < 0 \\ 0 & \text{otherwise} \end{cases} .$$

$\rightarrow$ The difference coefficients correspond to probing the process at different scales and locations, with n representing the location and j the scale.

• From the self-similarity of fractional Brownian motion, it follows that for $Y(t) = B_H(t)$:

$$E[d_j(n)^2] \propto 2^{j(2H+1)}.$$

Appendix: Scale spectrum

The scale spectrum of $\boldsymbol{X}$, relative to the Haar wavelet basis, is the sequence S_j defined by:

$$S_j = \frac{1}{2^{M-j}} \sum_{n=1}^{2^{M-j}} (d_j(n))^2 \ , \quad j = 1, 2, ..., M.$$

For fractional Brownian motion, the log of the scale spectrum is approximately *linear* in the scale j, with the slope determined by H.

Appendix: The relation to the scale spectrum

• If the underlying process has the correlation structure of fractional Brownian motion, the log-scale spectrum is affine in scale:

$$\mathbb{E}[\log(S_j)] = c_0(\sigma, H) + (2H+1)j.$$

Some issues:

- What are precision-of-parameter estimates?
- What is the role of the wavelets?
- What is the role of the scales and the shifts used?
- How should the regression be carried out?

Appendix: Analysis of precision

- Different wavelets (Daubechies) can be used as well.
- A detailed analysis of the biases and variances of the volatility and Hurst parameter estimators is possible when the underlying process is fractional Brownian motion.
- The decomposition with Haar wavelets gives the most efficient estimator (as long as H is below .7).

Appendix: On the parameter processes

Assume that the underlying process is fractional Brownian motion.

- It is possible to study the estimators $\hat{H}(c)$ and $\widehat{\log_2(\sigma^2)}(c)$ seen as processes indexed by the right end points c for moving time windows of equal length.
- The estimators $\hat{H}(c)$ and $\widehat{\log_2(\sigma^2)}(c)$, as processes indexed by c, have stationary Gaussian distributions with a covariance structure that is universal.

$\rightarrow$ It is possible to implement a filter to estimate more accurately the Hurst and volatility parameters.